

SUMS OF PRODUCTS OF POSITIVE OPERATORS AND SPECTRA OF LÜDERS OPERATORS

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ABSTRACT. Each bounded operator T on an infinite dimensional Hilbert space \mathcal{H} is a sum of three operators that are similar to positive operators; two such operators are sufficient if T is not a compact perturbation of a scalar. The spectra of Lüders operators (elementary operators on $B(\mathcal{H})$ with positive coefficients) of lengths at least three are not necessarily contained in \mathbb{R}^+ . On the other hand, the spectra of such operators of lengths (at most) two are contained in \mathbb{R}^+ if the coefficients on one side commute.

1. INTRODUCTION

Completely positive maps on $B(\mathcal{H})$ (the algebra of all bounded operators on a Hilbert space \mathcal{H}) of the form

$$(1.1) \quad \Psi(X) = \sum_{j=1}^n A_j^* X A_j,$$

have received a renewed interest recently especially in connection with quantum information theory (see [8], [9], [13], [18] and the references there). If all the coefficients A_j in (1.1) are positive operators such a map is called a Lüders operation. If n is finite then these are special cases of elementary operators, that is, maps of the form $X \mapsto \sum_{j=1}^n A_j X B_j$, whose spectra have been intensively studied in the past (see [5] and the references there), but only in the cases when both families of coefficients (A_j) and (B_j) are commutative. If \mathcal{H} is finite dimensional, then $B(\mathcal{H})$ is a Hilbert space for the inner product induced by the trace and it is easily verified that an elementary operator with positive coefficients A_j and B_j is a positive operator on this Hilbert space, so its spectrum is contained in $\mathbb{R}^+ := [0, \infty)$.

At the end of the paper [11] it was asked if the spectrum of a Lüders operator $X \mapsto \sum_{j=1}^n A_j X A_j$ with positive coefficients on $B(\mathcal{H})$ is necessarily contained in \mathbb{R}^+ if \mathcal{H} is infinite dimensional. We will show that, contrary to what one might expect, the answer to this question is negative. This will be a consequence of the fact that the operator $T = -1$ can be expressed as

$$(1.2) \quad T = \sum_{j=1}^n A_j B_j \quad \text{with positive } A_j, B_j \in B(\mathcal{H}).$$

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At first the author did not know how to prove that every operator $T \in B(\mathcal{H})$ is of the form (1.2), but then professor Heydar Radjavi told him that by [16] and [12] T is a sum of finitely many idempotents and, since every idempotent is similar to a projection, T is a sum of products of positive operators. To see this, note that an operator Q which is similar to a positive operator, say $Q = SPS^{-1}$, is a product of two positive operators: $Q = (SS^*)((S^{-1})^*PS^{-1})$. By Pearcy and Topping [12]) five idempotents are sufficient to express any T in this way and according to [19, Proposition 5.9] this is the minimal number since scalars are in general not sums of less than five idempotents. However, since idempotents are very special elements, we can not expect that 5 is the minimal n in (1.2).

One of the goals of this paper is to find the minimal n above. The result will imply that even the spectrum of a Lüders operator of small length is not necessarily contained in \mathbb{R}^+ . More precisely, in the next section we will show that every $T \in B(\mathcal{H})$ is a sum of three operators T_j each of which is similar to a positive operator. Moreover, if T is not a compact perturbation of a scalar, two operators T_j are sufficient. This result is optimal since compact perturbations of nonzero scalars can not be expressed in the form (1.2) with $n \leq 2$. We will also show that the trace class operators with trace not in \mathbb{R}^+ can not be expressed as $T_1 + T_2$ with both T_1 and T_2 similar to positive operators in $B(\mathcal{H})$. As a preliminary step in the proof of the main result we will first show that T is a sum of four operators T_j similar to positive ones, with some additional properties needed.

In the last section we will first apply this result to answer the above mentioned question from [11]. Then we will prove that the spectra of operators of the form $X \mapsto \sum_{j=1}^2 A_j X B_j$ with positive A_j and B_j are contained in \mathbb{R}^+ if $A_1 A_2 = A_2 A_1$ (or if $B_1 B_2 = B_2 B_1$).

Throughout the paper \mathcal{H} denotes an infinite dimensional separable Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . (The results hold also for non separable \mathcal{H} , but in their formulations the ideal of compact operators must be replaced by the unique proper maximal ideal of $B(\mathcal{H})$.) An operator $T \in B(\mathcal{H})$ is called positive if $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$ (thus T is not necessarily definite) and the set of all positive operators is denoted by $B(\mathcal{H})^+$.

2. SUMS OF OPERATORS SIMILAR TO POSITIVE OPERATORS

We begin with a simple and well-known observation. Let $S \in B(\mathcal{K} \oplus \mathcal{K})$ be a 2×2 operator matrix

$$(2.1) \quad S = \begin{bmatrix} u & x \\ y & z \end{bmatrix},$$

where u is invertible. Then S is invertible if and only if $z - yu^{-1}x$ is invertible and in this case

$$(2.2) \quad S^{-1} = \begin{bmatrix} u^{-1}(1 + xdyu^{-1}) & -u^{-1}xd \\ -dyu^{-1} & d \end{bmatrix}, \quad \text{where } d = (z - yu^{-1}x)^{-1}.$$

To prove this, multiply S from the left by the invertible matrix

$$\begin{bmatrix} u^{-1} & 0 \\ -yu^{-1} & 1 \end{bmatrix}$$

to obtain an upper-triangular matrix with 1 and $z - yu^{-1}x$ along the diagonal.

The main assertion of the following lemma can be deduced from the proof of Theorem 1 in [12], but later we will need some additional information from its proof in the form presented below.

Lemma 2.1. *Every operator $T \in B(\mathcal{H})$ is a sum of the form*

$$T = \sum_{j=1}^4 S_j T_j S_j^{-1},$$

where $S_j \in B(\mathcal{H})$ and the operators $T_j \in B(\mathcal{H})$ are positive with disjoint spectra $\sigma(T_j)$, each $\sigma(T_j)$ consists of at most two points, $\sigma(T_1) \subset [0, 1]$ and $\sigma(T_j) \subset (1, \infty)$ for $j \neq 1$. Moreover, the range of T_1 is closed and has infinite dimension and codimension.

In particular, T can be written as $T = \sum_{j=1}^4 A_j B_j$, where $A_j, B_j \in B(\mathcal{H})^+$.

Proof. Decompose \mathcal{H} into an orthogonal sum of two isomorphic closed subspaces, $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$; then T is represented by an operator matrix of the form

$$(2.3) \quad T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

It suffices to find diagonal positive operators $T_j = a_j \oplus b_j$ ($a_j, b_j \in B(\mathcal{K})$) and invertible operators S_j ($j = 1, 2, 3, 4$) of the form (2.1) such that $T = \sum_{j=1}^4 S_j T_j S_j^{-1}$. It turns out that we can even take S_j of the form

$$S_j = \begin{bmatrix} 1 & x_j \\ y_j & 1 + y_j x_j \end{bmatrix}.$$

Then

$$S_j T_j S_j^{-1} = \begin{bmatrix} a_j + s_j y_j & -s_j \\ y_j a_j - b_j y_j + y_j s_j y_j & b_j - y_j s_j \end{bmatrix}, \quad \text{where } s_j := a_j x_j - x_j b_j.$$

There are many appropriate choices for x_j, y_j, z_j, a_j, b_j in order to make the sum $\sum_{j=1}^4 S_j T_j S_j^{-1}$ equal to T . For example, if we let $y_1 = 0 = x_2$, $y_3 = 1$, $b_1 = 0$ and for $j \geq 2$ choose all a_j and b_j to be positive scalars with $a_j - b_j = 1$, and denote $\beta = \sum_{j=2}^4 b_j$ (so that $\sum_{j=2}^4 a_j = \beta + 3$), then

$$(2.4) \quad \sum_{j=1}^4 S_j T_j S_j^{-1} = \begin{bmatrix} a_1 + \beta + 3 + x_3 + x_4 y_4 & -a_1 x_1 - x_3 - x_4 \\ y_2 + x_3 + 1 + y_4 x_4 y_4 & \beta - x_3 - y_4 x_4 \end{bmatrix}.$$

To achieve that the matrix in (2.4) will be equal to T , we only need to choose x_3, x_4, y_4 in $B(\mathcal{K})$ and invertible $a_1 \in B(\mathcal{K})^+$ so that

$$(2.5) \quad a_1 + \beta + 3 + x_3 + x_4 y_4 = A \quad \text{and} \quad \beta - x_3 - y_4 x_4 = D,$$

for then the off-diagonal terms of the matrix (2.4) can be made equal to B and C by a suitable choice of y_2 and x_1 . Adding the two equations (2.5) we see, that we only need to choose x_4, y_4 and a_1 so that

$$(2.6) \quad x_4 y_4 - y_4 x_4 = A + D - a_1 - 2\beta - 3 =: T_0,$$

for then x_3 can be computed from either of the equations (2.5). So (for a fixed β), we first choose an invertible positive $a_1 \in B(\mathcal{K})$ of the form $\lambda + \mu p$, where $\lambda, \mu \in \mathbb{R}^+$ and p is a projection of infinite rank and nullity, such that $\sigma(a_1) \subset (0, 1]$ and T_0 is not a compact perturbation of a scalar. Then T_0 is a commutator by [2] (a simplified proof is in [1]), which means that there exist x_4 and y_4 satisfying

(2.6). By suitably choosing scalars a_j and b_j ($j \geq 2$) we can make the spectra of T_j disjoint for all j . \square

Remark 2.2. For a later use observe that in the above proof the spectra of a_j and b_j are disjoint for all j ; in fact all a_j and b_j chosen above are scalars, except possibly a_1 . Also note that the operator $S_1 T_1 S_1^{-1}$ has the form

$$\begin{bmatrix} a_1 & * \\ 0 & 0 \end{bmatrix},$$

where $a_1 \in B(\mathcal{K})^+$.

Theorem 2.3. *Every $T \in B(\mathcal{H})$ is of the form $T = \sum_{j=1}^3 S_j T_j S_j^{-1}$, where $S_j \in B(\mathcal{H})$ and the operators $T_j \in B(\mathcal{H})$ are positive (and invertible for $j \leq 2$) with finite spectra $\sigma(T_j)$, each $\sigma(T_j)$ consists of at most four points. Moreover, 0 is an isolated point of $\sigma(T_3)$, the range of T_3 is closed and has infinite dimension and codimension.*

Proof. As in the proof of Lemma 2.1 we represent T by the operator matrix (2.3). Now we try to find positive block-diagonal operators $T_j = a_j \oplus b_j$ and invertible operators $S_j \in B(\mathcal{H})$ of the form (2.1) (with $z - yu^{-1}x = 1$) such that $\sum_{j=1}^3 S_j T_j S_j^{-1} = T$. Denoting

$$S_j = \begin{bmatrix} u_j & x_j \\ y_j & z_j \end{bmatrix}, \text{ where } u_j \text{ is invertible and } z_j - y_j u_j^{-1} x_j = 1,$$

we compute (using (2.2)) that

$$S_j T_j S_j^{-1} = \begin{bmatrix} c_j + s_j v_j & -s_j \\ v_j c_j - b_j v_j + v_j s_j v_j & b_j - v_j s_j \end{bmatrix},$$

where

$$(2.7) \quad c_j := u_j a_j u_j^{-1}, \quad v_j := y_j u_j^{-1}, \quad \text{and} \quad s_j := c_j x_j - x_j b_j.$$

Note that if the spectra of b_j and c_j are disjoint, then from (2.7) a_j , y_j , b_j and x_j can all be computed from c_j , u_j , v_j , b_j , and s_j . (That the equation $c_j x_j - x_j b_j = s_j$ can be solved for x_j is Rosenblum's theorem [14, p. 8].) Further, we assume that the matrix S_3 is diagonal (that is, $x_3 = 0 = y_3$, so we will only need that the spectra of c_j and b_j are disjoint for $j = 1, 2$). Then the condition $\sum S_j T_j S_j^{-1} = T$ is equivalent to the following four equations:

$$(2.8) \quad s_1 v_1 + s_2 v_2 = A - c_1 - c_2 - c_3, \quad s_1 + s_2 = -B,$$

$$(2.9) \quad v_1 c_1 - b_1 v_1 + v_2 c_2 - b_2 v_2 + v_1 s_1 v_1 + v_2 s_2 v_2 = C, \quad v_1 s_1 + v_2 s_2 = -D + b_1 + b_2 + b_3.$$

Set $c := c_1 + c_2 + c_3$, $b := b_1 + b_2 + b_3$ and

$$s := s_1, \quad v := v_2, \quad w := v_2 - v_1.$$

Then from the second equation in (2.8) we get $s_2 = -(B + s)$; using this, the other three equations (2.8), (2.9) can be rewritten as

$$(2.10) \quad Bv + sw = c - A, \quad vB + ws = D - b,$$

$$(2.11) \quad v(c_1 + c_2 - sw) - (b_1 + b_2 + ws)v - wc_1 + b_1 w + wsw - vBv = C.$$

From (2.10) we have that $c_1 + c_2 - sw = A - c_3 + Bv$ and $b_1 + b_2 + ws = D - b_3 - vB$, hence (2.11) can be rewritten as

$$(2.12) \quad wsw - wc_1 + b_1w = C - v(A - c_3) + (D - b_3)v - vBv.$$

We are going to show that the system of equations (2.10), (2.12) has a solution.

First suppose that T is not a compact perturbation of a scalar. Then we may assume that in the matrix representation of T we have that $D = 0$ and that B is an isometry with the range of B isomorphic to its orthogonal complement in \mathcal{K} since by [2, Corollary 3.4] T is similar to such an operator. In this case we shall see that we can even afford to choose $s = 0$, so that the above system of equations simplifies to

$$(2.13) \quad Bv = c - A,$$

$$(2.14) \quad vB = -b,$$

$$(2.15) \quad b_1w - wc_1 = C - v(A - c_3) + (-b_3)v - vBv.$$

Since $B^*B = 1$, the equation (2.13) is equivalent to the following two:

$$(2.16) \quad v = B^*(c - A) \text{ and } P^\perp(c - A) = 0, \text{ where } P := BB^* \text{ and } P^\perp := 1 - P.$$

Using this expression for v , (2.14) can be rewritten as

$$(2.17) \quad b_1 + b_2 + b_3 = b = B^*(A - c)B.$$

If there exist v , c_j and b_j ($j = 1, 2, 3$) such that the equations (2.16) and (2.17) are satisfied and the spectra of c_1 and b_1 are disjoint, then the equation (2.15) can be solved for w by Rosenblum's theorem.

To show that the system (2.16), (2.17) has a solution, represent A by a 2×2 operator matrix with respect to the decomposition $\mathcal{K} = P\mathcal{K} \oplus P^\perp\mathcal{K}$. By Lemma 2.1 $A = \sum_{j=1}^4 A_j$ where each A_j is similar to a positive operator; moreover, by Remark 2.2 we may assume that (with respect to the decomposition $\mathcal{K} = P\mathcal{K} \oplus P^\perp\mathcal{K}$) A_4 is of the form

$$(2.18) \quad A_4 = \begin{bmatrix} a & r \\ 0 & 0 \end{bmatrix}, \text{ where } a \geq 0,$$

which means that $P^\perp A_4 = 0$. Thus, if we put $c_j = A_j$ for $j = 1, 2, 3$ (and $c = c_1 + c_2 + c_3$), then we have $P^\perp(A - c) = P^\perp A_4 = 0$, which is just the condition in (2.16). Further

$$(2.19) \quad B^*(A - c)B = B^*A_4B = B^*A_4PB = B^*GB,$$

where

$$G := A_4P = a \oplus 0.$$

Thus the operator $B^*(A - c)B$ is positive and hence it can be written (in many ways) as a sum of three positive operators b_j , which is just what the condition (2.17) requires. We may choose $b_3 = 0$. To see that it is possible to choose b_j and c_j ($j = 1, 2$) so that their spectra are disjoint, note that PB is a unitary operator from \mathcal{K} onto $P\mathcal{K}$ which intertwines a and $b = A - c$ by (2.19), hence b and a have the same spectrum. By Lemma 2.1 we may choose a and $c_j = A_j$ so that each of their spectra consists of at most two points, $\sigma(a) \subseteq (0, 1]$ and $\sigma(A_j) \subset (1, \infty)$ ($j = 1, 2, 3$). Since $b_j \geq 0$ and $b_1 + b_2 = b$, the spectra of b_j are contained in $[0, 1]$, hence $\sigma(b_j) \cap \sigma(c_j) = \emptyset$. Since $\sigma(b)$ consists of at most two points in $(0, 1]$, we may

choose b_1, b_2 to have the same property. (We may choose for b_1 a sufficiently small positive scalar, for example.)

Since T_j is similar to $a_j \oplus b_j$ and a_j is similar to $c_j = A_j$ ($j = 1, 2, 3$), $\sigma(T_j) = \sigma(A_j) \cup \sigma(b_j)$ consists of at most four points. Other properties of operators T_j stated in the theorem also follows easily from that of c_j and a_j chosen above.

Now we consider the case when T is a compact perturbation of a scalar. In this case let $E = 1 \oplus 0$, the projection onto the first summand in the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$. Then $\tilde{T} := T - E$ is not a compact perturbation of a scalar, so by the already proved case \tilde{T} can be expressed as $\tilde{T} = \sum_{j=1}^3 S_j(a_j \oplus b_j)S_j^{-1}$, where $a_j \geq 0$ and $b_j \geq 0$ and S_3 is block-diagonal. Since S_3 commutes with E , we have

$$T = \tilde{T} + E = \sum_{j=1}^2 S_j(a_j \oplus b_j)S_j^{-1} + S_3((a_3 \oplus b_3) + E)S_3^{-1},$$

which is a sum of three operators similar to positive ones with (at most) four-point spectra. \square

Remark 2.4. Observe that in the proof of Theorem 2.3 the operator T_3 is of the form $e \oplus 0$, where e is similar to a positive invertible operator with at most two-point spectrum.

Corollary 2.5. *Each $T \in B(\mathcal{H})$ can be expressed as $T = \sum_{j=1}^3 A_j B_j$, where $A_j, B_j \in B(\mathcal{H})^+$.*

Theorem 2.6. *If $T \in B(\mathcal{H})$ is not a compact perturbation of a scalar, then T is a sum of two operators similar to positive operators.*

Proof. We have to show that in the proof of Theorem 2.3 a_3 and b_3 can be taken to be 0. That b_3 can be taken to be 0 has been already observed in that proof. Now note that in the matrix representation (2.3) of T we may assume, in addition to $D = 0$ and B is an isometry, that A is not a compact perturbation of a scalar. For this, we simply decompose the second copy of \mathcal{K} into two orthogonal isomorphic closed subspaces, $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$, and decompose \mathcal{H} as $\mathcal{H} = \mathcal{K}_1^\perp \oplus \mathcal{K}_1$. Since B maps \mathcal{K}_1 isometrically into \mathcal{K}_1^\perp the matrix of T has 0 on the (2,2) position and an isometry with infinitely codimensional range on the (1,2) position. The new element on the position (1,1) is then not a compact perturbation of a scalar. So we will assume that already in the initial matrix representation of T the element A is not a compact perturbation of a scalar. Consider now the matrix of A relative to the decomposition of the Hilbert space of A into the range of B and its orthogonal complement. Since A is not a compact perturbation of a scalar, by Theorem 2.3 and Remark 2.4 A is of the form $A = \sum_{j=1}^3 \tilde{A}_j$, where \tilde{A}_1 and \tilde{A}_2 are similar to positive invertible operators each with at most four points in its spectrum and \tilde{A}_3 is of the form $e \oplus 0$ with e similar to a positive invertible operator with a two-point spectrum. By the same reasoning as in the proof of Theorem 2.3 (see the paragraph containing (2.18); the role of A_4 is now played by \tilde{A}_3) we see that the system of equations (2.16), (2.17) has a solution such that $c_j = \tilde{A}_j$ for $j = 1, 2$ and $c_3 = 0 = b_3 = 0$. But we have to show also that we can achieve $\sigma(c_j) \cap \sigma(b_j) = \emptyset$ ($j = 1, 2$) in order to assure that (2.15) has a solution for w and that x_j can be computed from the last equation in (2.7). For this we note now that the operator $B^*(A - c)B = B^*\tilde{A}_3B$ is unitarily equivalent to e . Since $\sigma(c_j)$ ($j = 1, 2$) is a finite subset of $(0, \infty)$ and $\sigma(B^*(A - c)B)$ consists of just two positive points, it

follows that $B^*(A - c)B$ can be expressed as a sum $b_1 + b_2$, where $b_j \geq 0$ and $\sigma(b_j) \cap \sigma(c_j) = \emptyset$ for both $j = 1, 2$. \square

An operator $T \in \mathcal{B}(\mathcal{H})$ of the form $\lambda + K$, where $\lambda \in \mathbb{C} \setminus \mathbb{R}^+$ and K is compact, is not of the form

$$(2.20) \quad PQ + RS \quad \text{for any } P, Q, R, S \in \mathcal{B}(\mathcal{H})^+.$$

To see this, just note that the spectrum of the coset $\dot{R}\dot{S}$ in the Calkin algebra is the same as the spectrum of $\dot{S}^{1/2}\dot{R}\dot{S}^{1/2}$, hence contained in \mathbb{R}^+ , while the spectrum of $\lambda - \dot{P}\dot{Q}$ is contained in the ray $\lambda - \mathbb{R}^+$ which is disjoint with \mathbb{R}^+ .

Each compact operator on a Hilbert space is an additive commutator of two bounded operators [1]. By an analogy one might conjecture that each compact operator is a sum of two operators similar to positive ones, but this is not true.

Proposition 2.7. *If $T \in \mathcal{C}^1(\mathcal{H})$ (the trace class) is nonzero and $\text{Tr}(T)$ is not positive, then T is not a sum of two operators in $\mathcal{B}(\mathcal{H})$ similar to positive ones.*

Proof. Assume the contrary, that $T = S_1 A S_1^{-1} + S_2 B S_2^{-1}$, where $A, B \in \mathcal{B}(\mathcal{H})^+$. Put $F := -S_1^{-1} T S_1$ and $S = S_1^{-1} S_2$. Then

$$(2.21) \quad F + A = -S B S^{-1}.$$

Considering the essential spectra, it follows from (2.21) and the positivity of A and B that A and B must be compact. We claim, that A and B must be in the Hilbert-Schmidt class $\mathcal{C}^2(\mathcal{H})$. For a proof we may first replace B by a unitarily equivalent operator (and modify S accordingly) to reduce to the situation when A and B can be diagonalized in the same orthonormal basis \mathbb{B} of \mathcal{H} . Let (α_j) and (β_j) be the lists of eigenvalues of A and B in decreasing order (each eigenvalue repeated according to its multiplicity). From (2.21) we have $AS + SB = G$, where $G := -FS$. Denoting by $\sigma_{i,j}$ and $\psi_{i,j}$ the entries of the matrices of S and G in the basis \mathbb{B} , this means that

$$(2.22) \quad (\alpha_i + \beta_j) \sigma_{i,j} = \psi_{i,j}.$$

Let $\gamma_j := (\sum_i |\psi_{i,j}|^2)^{1/2}$ and note that $\sum_j \gamma_j^2 < \infty$ since $G \in \mathcal{C}^2(\mathcal{H})$. Since S is invertible (in particular, bounded from below), there exists a scalar $\gamma > 0$ such that $\sum_i |\sigma_{i,j}|^2 \geq \gamma$ for all j , hence (2.22) implies that

$$\beta_j^{-2} \gamma_j^2 = \beta_j^{-2} \sum_i |\psi_{i,j}|^2 = \sum_i \frac{(\alpha_i + \beta_j)^2}{\beta_j^2} |\sigma_{i,j}|^2 \geq \sum_i |\sigma_{i,j}|^2 \geq \gamma,$$

whenever $\beta_j \neq 0$. Thus $\beta_j^2 \leq |\gamma_j|^2 \gamma^{-1}$ and consequently $\sum_j \beta_j^2 < \infty$, which means that $B \in \mathcal{C}^2(\mathcal{H})$. Similarly (or from (2.21), since $F \in \mathcal{C}^2(\mathcal{H})$) we see that $A \in \mathcal{C}^2(\mathcal{H})$.

By considering the polar decomposition of S of the form $S = RU$, where R is positive and U is unitary, we may rewrite (2.21) in the form

$$(2.23) \quad F + A = -RCR^{-1},$$

where $C := UBU^* \geq 0$. Assume for a moment that in some orthonormal basis of \mathcal{H} the operator R can be represented by a diagonal matrix and let $[\alpha_{i,j}]$, $[\phi_{i,j}]$ and

$[\gamma_{i,j}]$ be the matrices of A , F and C (respectively) in this basis. Then, considering the sums of diagonal terms of matrices, (2.23) implies that

$$(2.24) \quad \sum_{j=1}^n \psi_{j,j} + \sum_{j=1}^n \alpha_{j,j} = - \sum_{j=1}^n \gamma_{j,j}.$$

Letting $n \rightarrow \infty$, the first sum in (2.24) tends to $Tr(F) = Tr(T) \in \mathbb{C} \setminus (0, \infty)$, while the second and the third sums converge to elements in $[0, \infty]$. This shows that the equality (2.24) can hold for all n only if $Tr(T) = 0$ and $\psi_{j,j} = 0 = \alpha_{j,j}$ for all j . Since $A \in B(\mathcal{H})^+$, the condition $\alpha_{j,j} = 0$ for all j implies that $A = 0$. But then B is similar to T , hence $Tr(B) = 0$, which implies (since $B \geq 0$) that $B = 0$. In this case $T = 0$, which was excluded by the hypothesis of the proposition. Now we will show by an approximation argument that (2.23) leads to a contradiction even if R can not be diagonalized.

By the Weyl - von Neumann theorem [4, p. 214], given $\varepsilon > 0$, there exist a diagonal hermitian operator D and an operator $H \in C^2(\mathcal{H})$ with $\|H\|_2 < \varepsilon$ (where $\|\cdot\|_2$ denotes the Hilbert - Schmidt norm) such that $R = D + H$. If ε is small enough then D is invertible (since $D = R - H = R(1 - R^{-1}H)$) and

$$\|D^{-1}\| \leq \|R^{-1}\| \sum_{n=0}^{\infty} \|R^{-1}H\|^n \leq \frac{\|R^{-1}\|}{1 - \varepsilon\|R^{-1}\|}.$$

Further, if ε is small enough then $1 + HD^{-1}$ is invertible and

$$RCR^{-1} = (1 + HD^{-1})DCD^{-1}(1 + HD^{-1})^{-1}.$$

Since $(1 + HD^{-1})^{-1} = 1 - HD^{-1}(1 + HD^{-1})^{-1}$, we may write

$$\begin{aligned} RCR^{-1} &= DCD^{-1} - DCD^{-1}HD^{-1}(1 + HD^{-1})^{-1} \\ &\quad + HCD^{-1}[1 - HD^{-1}(1 + HD^{-1})^{-1}], \end{aligned}$$

hence (since B and therefore also C is in $C^2(\mathcal{H})$ by the first paragraph of this proof)

$$\begin{aligned} \|RCR^{-1} - DCD^{-1}\|_1 &\leq \|H\|_2\|C\|_2\|D^{-1}\| \cdot \\ &\quad [\|D\|\|D^{-1}\|\|(1 + HD^{-1})^{-1}\| + \|(1 - HD^{-1}(1 + HD^{-1})^{-1})\|]. \end{aligned}$$

It follows that $\|RCR^{-1} - DCD^{-1}\|_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$. This allows us to conclude in essentially the same way as in the previous paragraph (by considering the sums of diagonal entries of matrices) that (2.23) leads to a contradiction. \square

For most of the above proof it would be sufficient if we assumed that $T \in C^2(\mathcal{H})$ (instead of $T \in C^1(\mathcal{H})$), but the problem is that for an operator T not in $C^1(\mathcal{H})$ the sum of diagonal entries of its matrix relative to a general orthogonal basis can be quite arbitrary (it need not even be defined [7]).

Problem. Which compact operators on an infinite dimensional Hilbert space can be written as $T_1 + T_2$, where T_1 and T_2 are similar to positive operators?

Theorem 2.6 implies that all operators can be approximated in norm by sums of two operators similar to positive ones; but concerning such approximation a much stronger result holds: it follows from [6, Theorem 3.10] that both summands can be taken to be similar to the same positive operator.

3. ON SPECTRA OF LÜDERS OPERATORS

For two commutative m -tuples (A_j) and (B_j) of elements of $B(\mathcal{H})$ the spectrum $\sigma(\Phi)$ of the map $\Phi(X) := \sum_{j=1}^m A_j X B_j$ on $B(\mathcal{H})$ can be described in terms of spectra of (A_j) and (B_j) ([5], [11]); in particular $\sigma(\Phi) \subseteq \mathbb{R}^+$ if $A_j, B_j \in B(\mathcal{H})^+$. For noncommutative (A_j) and (B_j) the situation may be completely different. One consequence of Theorem 2.3 is that for an infinite dimensional Hilbert space \mathcal{H} the spectra of Lüders operators on $B(\mathcal{H})$ are not necessarily contained in \mathbb{R}^+ .

Proposition 3.1. *Let \mathcal{H} be an infinite dimensional Hilbert space. Every complex number λ can be an eigenvalue of a Lüders operator on $B(\mathcal{H})$ of length 3 (or more).*

Proof. Decompose \mathcal{H} as $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$. By Corollary 2.5 there exist $A_j, B_j \in B(\mathcal{K})^+$ such that $\sum_{j=1}^3 A_j B_j = \lambda$. By a simple calculation this implies that the operator

$$X_0 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue λ of the Lüders operator Φ on $B(\mathcal{H})$ defined by $\Phi(X) = \sum_{j=1}^3 T_j X T_j$, where

$$T_j = \begin{bmatrix} A_j & 0 \\ 0 & B_j \end{bmatrix}.$$

□

Theorem 3.2. *Suppose that $A_j, B_j \in B(\mathcal{H})^+$ ($j = 1, 2$) and let Φ be the map on $B(\mathcal{H})$ defined by $\Phi(X) = \sum_{j=1}^2 A_j X B_j$. If $A_1 A_2 = A_2 A_1$ (or if $B_1 B_2 = B_2 B_1$) then the spectrum of Φ is contained in \mathbb{R}^+ .*

Proof. Since boundary points of the spectrum of any operator are approximate eigenvalues [3], it suffices to show that each approximate eigenvalue λ of Φ is in \mathbb{R}^+ . By considering the space $\mathcal{B} := \ell_\infty(B(\mathcal{H}))/c_0(B(\mathcal{H}))$, where $\ell_\infty(B(\mathcal{H}))$ is the space of all bounded sequences with the entries in $B(\mathcal{H})$ and $c_0(B(\mathcal{H}))$ is the subspace of all sequences converging (in norm) to 0, we may reduce the approximate eigenvalues of Φ to proper eigenvalues of the corresponding operator $\tilde{\Phi}$ on \mathcal{B} . Here of course $\tilde{\Phi}$ is defined by $\tilde{\Phi}([X_n]) = [\Phi(X_n)]$, where $[X_n]$ denotes the coset of a sequence $(X_n) \in \ell_\infty(B(\mathcal{H}))$. Note that $\tilde{\Phi}$ is again an elementary operator, namely of the form

$$(3.1) \quad \tilde{\Phi}(Y) = \sum_{j=1}^2 \tilde{A}_j Y \tilde{B}_j \quad (Y \in \mathcal{B}),$$

where \tilde{A} denotes the coset in \mathcal{B} of the constant sequence $(A, A, \dots) \in \ell_\infty(B(\mathcal{H}))$ for each $A \in B(\mathcal{H})$. Since \mathcal{B} is a C^* -algebra, we can regard it as a subalgebra of $B(\mathcal{K})$ for some (non-separable) Hilbert space \mathcal{K} and by the formula (3.1) we may regard the map $\tilde{\Phi}$ to be defined on all $B(\mathcal{K})$. Any approximate eigenvalue λ of Φ is then an eigenvalue of $\tilde{\Phi}$. Choose a nonzero eigenvector Y corresponding to λ . \mathcal{K} is not separable, but it can be expressed as an orthogonal sum of separable subspaces \mathcal{K}_i that reduce all the operators A_j, B_j and Y . If i is such that $Y|_{\mathcal{K}_i} \neq 0$, then λ is an eigenvalue of the operator Ψ on $B(\mathcal{K}_i)$ defined by $\Psi(X) = \sum_{j=1}^2 C_j X D_j$, where $C_j = A_j|_{\mathcal{K}_i}$ and $D_j = B_j|_{\mathcal{K}_i}$. So it suffices to show that all eigenvalues of such

operators are in \mathbb{R}^+ . Thus, (adapting the notation) we may assume that λ is an eigenvalue of Φ . Denote by X a corresponding eigenvector with $\|X\| = 1$, hence

$$(3.2) \quad \sum_{j=1}^2 A_j X B_j = \lambda X.$$

Suppose that A_1 and A_2 commute. Then by Voiculescu's version [17] of the Weyl-von Neumann-Berg theorem, given $\varepsilon > 0$, there exist commuting diagonal hermitian operators $C_j \in \mathcal{B}(\mathcal{H})$ and Hilbert-Schmidt operators $H_j \in \mathcal{C}^2(\mathcal{H})$ such that $A_j = C_j + H_j$ and $\|H_j\|_2 < \varepsilon$ ($j = 1, 2$). Let $C_j = C_j^+ - C_j^-$ be the decomposition of C_j into the positive and the negative part and denote by Q_j the range projection of C_j^- . Then $A_j + C_j^- = C_j^+ + H_j$, hence (since $Q_j C_j^+ = 0$ and $Q_j C_j^- = C_j^-$)

$$Q_j A_j Q_j + C_j^- = Q_j H_j Q_j \in \mathcal{C}^2(\mathcal{H}).$$

This implies that $C_j^- \in \mathcal{C}^2(\mathcal{H})$ and $\|C_j^-\|_2 \leq \|H_j\|_2 < \varepsilon$. So, replacing C_j by C_j^+ and H_j by $H_j - C_j^-$ (and the initial ε by $\varepsilon/2$), we may assume that $C_j \geq 0$. Let P be any finite rank projection that commutes with C_1 and C_2 . (Note that, since C_1, C_2 are commuting diagonal operators, there exist a net of such projections P converging strongly to the identity.) From (3.2) we have that $\sum P A_j X B_j X^* P = \lambda P X X^* P$, hence applying the trace Tr we obtain

$$(3.3) \quad \sum_{j=1}^2 (Tr(PC_j X B_j X^* P) + Tr(PH_j X B_j X^* P)) = \lambda Tr(P X X^* P).$$

Since P commutes with C_j ,

$$(3.4) \quad tr(PC_j X B_j X^* P) = Tr(C_j P X B_j X^* P) = Tr(C_j^{1/2} P X B_j X^* P C_j^{1/2}) \geq 0.$$

Further (since $\|Z\|_2 = \|Z^*\|_2$ for all $Z \in \mathcal{B}(\mathcal{H})$),

$$(3.5) \quad |Tr(PH_j X B_j X^* P)| \leq \|H_j\|_2 \|X B_j X^* P\|_2 = \|H_j\|_2 \|P X B_j X^*\|_2 < \varepsilon \|P X\|_2,$$

where we have assumed (without loss of generality) that $\|B_j\| \leq 1$. If P is sufficiently close to 1 so that $PX \neq 0$, then from (3.3) and (3.5) we have that

$$\left| \lambda - \sum_{j=1}^2 \frac{Tr(PC_j X B_j X^* P)}{Tr(P X X^* P)} \right| \leq \varepsilon \sum_{j=1}^2 \frac{\|P X\|_2}{Tr(P X X^* P)} = \frac{2\varepsilon}{\|P X\|_2}.$$

Letting in this estimate $P \rightarrow 1$, $\varepsilon \rightarrow 0$ and using (3.4), we see that $\lambda \geq 0$. \square

Remark 3.3. Theorem 3.2 can be extended to operators of the form

$$(3.6) \quad X \mapsto \sum_{j=1}^n A_j X B_j$$

if the coefficients on one side, say all the A_j , are smooth nonnegative functions $A_j = f_j(H_1, H_2)$ of a pair of commuting hermitian operators (H_1, H_2) . Namely, in this case it can be shown (using the Fourier transform) that small Hilbert-Schmidt perturbations of (H_1, H_2) result in small Hilbert-Schmidt perturbations of $f_j(H_1, H_2)$. The author does not know if the theorem can be extended to the general situation, when all the A_j commute, but the B_j do not necessarily commute.

Problems. 1. Can Theorem 3.2 be generalized to operators of length greater than 2?

2. Suppose that all A_j, B_j are positive and for each j at least one of A_j, B_j is compact. Then it can be deduced from [15, Corollary 6.6] (see [10]) that all eigenvalues of the operator (3.6) are contained in \mathbb{R}^+ . Is the same true for the entire spectrum?

3. Can in Theorem 3.2 the commutativity condition be replaced by commutativity modulo compact operators?

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